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Topic 1

Matrix Fundamentals & Operations

What You Will Learn in Topic 1

By the end of this topic, you will be able to:

- Name the dimensions of any matrix and identify any entry using subscript notation (e.g. a_{23})
- Add and subtract matrices by combining matching entries
- Multiply a matrix by a scalar (single number)
- Recognise and use the zero matrix and identity matrix

1.1 What Is a Matrix?

• Definition: Matrix

A **matrix** is a rectangular array of numbers arranged in rows and columns. We write a matrix inside square brackets.

A matrix with m rows and n columns is called an $m \times n$ **matrix** (read "m by n").

Consider a store that sells three product types in two locations. The inventory data can be organized as a matrix:

$$\begin{array}{ll} \mathbf{I} = \begin{bmatrix} 120 & 85 & 64 \\ 95 & 110 & 73 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Store 1} \\ \leftarrow \text{Store 2} \end{array} \\ \text{Product AProduct BProduct C} & \end{array}$$

1.2 Dimensions & Entry Notation

We denote a matrix by a capital letter (e.g. A) and write its dimensions as $\mathbf{A}_{\mathbf{m \times n}}$, where m = number of rows and n = number of columns.

• Entry Notation

The entry in row i, column j of matrix A is written $\mathbf{a}_{\mathbf{ij}}$ (read "a sub i-j").

Row index i counts from top (1, 2, 3, …). Column index j counts from left (1, 2, 3, …).

Guided Note: A matrix with 3 rows and 4 columns is a __3×4__ matrix. The entry in the 2nd row, 3rd column of matrix *B* is written __b_{23}__.

• Example 1: Identifying Dimensions & Entries

Given the matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & -5 \end{bmatrix}$$

Step 1 — Dimensions: A has 2 rows and 3 columns, so A is a **2 × 3 matrix**.

Step 2 — Identify entries by position:

$a_{11} = 3 a_{12} = -1 a_{13} = 4$

$a_{21} = 2 a_{22} = 0 a_{23} = -5$

1.3 Matrix Addition & Subtraction

• Addition & Subtraction Rule

Two matrices can be added (or subtracted) **only if they have the same dimensions**.

To add (subtract), combine corresponding entries:

C = A + B means $\mathbf{c_{ij} = a_{ij} + b_{ij}}$

C = A − B means $\mathbf{c_{ij} = a_{ij} - b_{ij}}$

• Example 2: Matrix Addition

Find A + B, where:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}$$

Step 1 — Check dimensions: Both A and B are 2×2. [OK] Same dimensions.

Step 2 — Add each corresponding pair:

$c_{11} = 1 + 5 = 6c_{12} = 3 + (-1) = 2$

$c_{21} = -2 + 3 = 1c_{22} = 4 + 2 = 6$

Step 3 — Write the result:

$$\mathbf{A + B} = \begin{bmatrix} 6 & 2 \\ 1 & 6 \end{bmatrix}$$

• Example 3: Matrix Subtraction

Find C − D, where:

$$\mathbf{C} = \begin{bmatrix} 7 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix}$$

Step 1 — Check dimensions: Both are 2×2. [OK]

Step 2 — Subtract corresponding entries:

$c_{11} - d_{11} = 7 - 3 = 4c_{12} - d_{12} = 2 - 5 = -3$

$c_{21} - d_{21} = -1 - 2 = -3c_{22} - d_{22} = 3 - (-4) = 7$

Step 3 — Write the result:

$$\mathbf{C - D} = \begin{bmatrix} 4 & -3 \\ -3 & 7 \end{bmatrix}$$

!! Dimension Mismatch!

If two matrices have different dimensions, their sum or difference is **undefined**. For example, a 2×3 matrix cannot be added to a 2×2 matrix.

1.4 Scalar Multiplication

• Scalar Multiplication

A **scalar** is a single number (not a matrix). To multiply a matrix by a scalar k, multiply **every entry** by k:

kA means $(kA)_{ij} = k \cdot a_{ij}$

• Example 4: Scalar Multiplication

Find 3A, where:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

Step 1 — Multiply each entry by 3:

$3 \cdot 2 = 63 \cdot (-3) = -9$

$3 \cdot 1 = 33 \cdot 4 = 12$

Step 2 — Write the result:

$$3A = \begin{bmatrix} 6 & -9 \\ 3 & 12 \end{bmatrix}$$

• Example 5: Combined Operations: 2A − 3B

Find 2A − 3B, where:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$$

Step 1 — Compute 2A:

$$\mathbf{2A} = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}$$

Step 2 — Compute 3B:

$$\mathbf{3B} = \begin{bmatrix} -3 & 12 \\ 6 & 3 \end{bmatrix}$$

Step 3 — Subtract: 2A − 3B:

2 − (−3) = 54 − 12 = −8

6 − 6 = 00 − 3 = −3

$$\mathbf{2A - 3B} = \begin{bmatrix} 5 & -8 \\ 0 & -3 \end{bmatrix}$$

Verification: Check entry (1,1): 2(1) − 3(−1) = 2 + 3 = 5 [OK]

1.5 Zero & Identity Matrices

• Zero Matrix (O)

The **zero matrix** $O_{m\times n}$ has every entry equal to 0. It is the additive identity:

A + O = O + A = A (for any matrix A of the same dimensions)

• Identity Matrix (I_n)

The **identity matrix** I_n is an n×n square matrix with 1s on the main diagonal and 0s everywhere else.

It is the multiplicative identity: **A · I = I · A = A** (for any n×n matrix A).

The 2×2 and 3×3 identity matrices:

$$\mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• Example 6: Properties of Zero & Identity Matrices

Given:

$$\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \quad \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Part A: Find A + O

$4+0=4$, $-2+0=-2$, $3+0=3$, $1+0=1$

$$\mathbf{A + O} = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Result is A itself, confirming A + O = A. [OK]

Part B: Find A · I (Preview of Topic 2)

Row 1 · Col 1: $(4)(1) + (-2)(0) = 4$

Row 1 · Col 2: $(4)(0) + (-2)(1) = -2$

Row 2 · Col 1: $(3)(1) + (1)(0) = 3$

Row 2 · Col 2: $(3)(0) + (1)(1) = 1$

$$\mathbf{A \cdot I} = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Result is A itself, confirming A · I = A. [OK]

Topic 1 Guided Practice

1. State the dimensions of a matrix with 4 rows and 2 columns.
2. If M is a 3×3 matrix, write the notation for the entry in row 1, column 3.
3. Find the sum:

$$\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}$$

4. Find 4A, where A =

$$\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$$

5. Find 2X − Y, where:

$$\mathbf{X} = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$$

• Key Takeaways from Topic 1

• Matrices must have the **same dimensions** to be added or subtracted.

• Scalar multiplication distributes across the matrix: k(A + B) = kA + kB.

• The zero matrix O acts like the number 0 for addition.

• The identity matrix I acts like the number 1 for multiplication.

Topic 1 Quick-Reference Summary

Operation	Requirement	How To Apply	Example
Addition A+B	Same m×n dimensions	Add matching entries: cij = aij + bij	[1,2]+[3,4]=[4,6]
Subtraction A-B	Same m×n dimensions	Subtract matching entries: cij = aij - bij	[5,3]-[2,1]=[3,2]
Scalar kA	Any matrix	Multiply EVERY entry by k	3*[2,-1]=[6,-3]
Zero Matrix O	Same size as A	A + O = A (additive identity)	Any A + O = A
Identity I	Square matrices only	A*I = I*A = A (multiplicative identity)	Any square A

Topic 4 Guided Practice

1. Write the system as a matrix equation $AX = B$:

$4x - y = 5$

$2x + 3y = 7$

2. Solve the system using Cramer's Rule:

$x + 5y = 3$

$2x - y = 4$

3. Solve the system using Gaussian Elimination (RREF):

$x + 2y + z = 3$

$2x - y + z = 0$

$3x + y + 2z = 3$

• Key Takeaways from Topic 4

• $AX = B$ compresses an entire system into three matrices: Coefficients, Variables, Constants.

• If A is invertible, $\mathbf{X = A^{-1}B}$ provides the exact unique solution.

• **Cramer's Rule** is a determinant shortcut: $x_i = \det(A_i) / \det(A)$.

• **Gaussian Elimination** is the most robust method; it works even when A is not invertible.

• Always look at the RREF to identify if a system is Independent, Inconsistent, or Dependent.

Key Formulas — Topic 4

Matrix Equation: $AX = B$

Inverse Method: $X = A^{-1}B$ (multiply A^{-1} on the LEFT)

Cramer's Rule: $x_i = \det(A_i) / \det(A)$ where A_i replaces column i with B

Topic 4 Quick-Reference: Choosing a Solution Method

Method	Best Used When	Key Steps	Singular A?

Inverse: X = A-1B	Small system; A is invertible	1. Find A-1 2. Multiply: X = A-1B	No (need det ≠ 0)
Cramer's Rule	2x2 or 3x3; quick scalar answer	1. Find det(A) 2. Replace col i of A with B to get Ai 3. xi = det(Ai) / det(A)	No (need det ≠ 0)
Gaussian Elimination / RREF	Any system, any size	1. Form [A|B] 2. Row-reduce to RREF 3. Read solution or identify type	Yes — handles all cases

!! !! Common Mistakes — Topic 4

Mistake 1 — Multiplying A^{-1} on the wrong side: From AX = B, the correct move is $A^{-1}(AX) = A^{-1}B$, giving $X = A^{-1}B$. Writing $XA^{-1} = BA^{-1}$ is incorrect because matrix multiplication is not commutative.

Mistake 2 — Replacing the wrong column in Cramer's Rule: To find x_1, replace COLUMN 1 of A with vector B. To find x_2, replace COLUMN 2. Replacing the wrong column gives a completely wrong answer.

Mistake 3 — Row operation applied to wrong row in RREF: Write each step clearly: "$R_2 + 3R_1 \rightarrow R_2$" changes ONLY R_2. Accidentally changing R_1 as well corrupts all subsequent steps.

Mistake 4 — Misreading RREF for system type: A row [0 0 0 | 0] means the system is DEPENDENT (free variable, infinitely many solutions). A row [0 0 0 | c] with c ≠ 0 means INCONSISTENT (no solution). These two look similar — check the last column carefully.

gallons leave. Write the system of equations.

2. Product X requires 2 units of Wood and 1 unit of Metal. Product Y requires 3 units of Wood and 4 units of Metal. Wood costs $5/unit, Metal costs $8/unit. Use matrix multiplication to find the total cost of each product.

• Key Takeaways from Topic 5

• Matrices elegantly condense complex real-world systems (like traffic or economics) into solvable equations $AX = B$.

• **Inventory/Cost Analysis** uses matrix multiplication to link requirements to prices.

• **Leontief Models** use inverse matrices ($X = (I-A)^{-1}D$) to find total production for an economy.

• **Geometric Transformations** use matrix multiplication to alter coordinates, reading right-to-left.

Key Formulas — Topic 5

Leontief Model: $X = AX + D \rightarrow X = (I - A)^{-1}D$

Transformation Composition: "Apply T_1 first, then T_2" = $T_2 \cdot T_1$ (T_1 on the right)

!! !! Common Mistakes — Topic 5

Mistake 1 — Transformation order (right-to-left rule): "Apply rotation R first, then reflection F" is written as the product $F \cdot R$ — the FIRST transformation applied goes on the RIGHT. Writing $R \cdot F$ gives the opposite composition and a completely different result.

Mistake 2 — Leontief: using A instead of (I − A): The total output formula is $X = (I - A)^{-1}D$. You must first compute $I - A$, then find its inverse. Using A^{-1} directly is incorrect.

Mistake 3 — Network flow: wrong direction for equations: Conservation of flow requires Flow IN = Flow OUT at every node. Carefully identify which flows enter and which exit each node before writing the equation.

Mistake 4 — Cost matrix: wrong multiplication order: If P is (products × materials) and C is (materials × stores), then PC gives cost per product per store. Computing CP gives a dimensionally different result with no practical meaning.

Mistake 5 — Leontief: not solving (I − A)D instead of (I − A)⁻¹D: The formula requires the INVERSE of $(I - A)$ multiplied by D, not just $(I - A)$ multiplied by D. Always find the inverse of $(I - A)$ first.

Practice & Assessment

Matrix Operations Drill Set

Complete the following matrix operations. Show all intermediate steps.

Problem 1: Find A + B.

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$$

Solution 1:

Step 1: Check dimensions. Both are 2×2. [OK]

Step 2: Add corresponding entries.

$c_{11} = 4 + 3 = 7$, $c_{12} = -1 + 2 = 1$

$c_{21} = 2 + (-1) = 1$, $c_{22} = 5 + 4 = 9$

$$\mathbf{A + B} = \begin{bmatrix} 7 & 1 \\ 1 & 9 \end{bmatrix}$$

Problem 2: Find C - D.

$$\mathbf{C} = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} -4 & 2 \\ 6 & -1 \end{bmatrix}$$

Solution 2:

Step 1: Check dimensions. Both are 2×2. [OK]

Step 2: Subtract corresponding entries.

$c_{11} = -2 - (-4) = 2$, $c_{12} = 5 - 2 = 3$

$c_{21} = 0 - 6 = -6$, $c_{22} = 3 - (-1) = 4$

$$\mathbf{C - D} = \begin{bmatrix} 2 & 3 \\ -6 & 4 \end{bmatrix}$$

Problem 3: Find E + F.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -4 & 0 \end{bmatrix}$$

Solution 3:

Step 1: Check dimensions. Both are 2×3. [OK]

Step 2: Add corresponding entries.

Row 1: 1+(−1)=0, 0+2=2, −3+3=0

Row 2: 2+5=7, 4+(−4)=0, 1+0=1

$$\mathbf{E + F} = \begin{bmatrix} 0 & 2 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

Problem 4: Find G - H.

$$\mathbf{G} = \begin{bmatrix} 8 & 2 \\ -3 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} 5 & -2 \\ 3 & 1 \\ -4 & 2 \end{bmatrix}$$

Solution 4:

Step 1: Check dimensions. Both are 3×2. [OK]

Step 2: Subtract corresponding entries.

Row 1: 8−5=3, 2−(−2)=4

Row 2: −3−3=−6, 1−1=0

Row 3: 0−(−4)=4, 5−2=3

$$\mathbf{G - H} = \begin{bmatrix} 3 & 4 \\ -6 & 0 \\ 4 & 3 \end{bmatrix}$$

Problem 5: Find 3A.

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$$

Solution 5:

Step 1: Multiply every entry by 3.

$3(2)=6$, $3(-4)=-12$, $3(0)=0$, $3(5)=15$

$$\mathbf{3A} = \begin{bmatrix} 6 & -12 \\ 0 & 15 \end{bmatrix}$$

Problem 6: Find 2X - Y.

$$\mathbf{X} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$$

Solution 6:

Step 1: Compute 2X.

2X = [2, −6 ; 8, 4]

Step 2: Compute 2X − Y.

$c_{11} = 2 - (-2) = 4$, $c_{12} = -6 - 1 = -7$

$c_{21} = 8 - 0 = 8$, $c_{22} = 4 - 5 = -1$

$$\mathbf{2X - Y} = \begin{bmatrix} 4 & -7 \\ 8 & -1 \end{bmatrix}$$

Problem 7: Find AB.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution 7:

Step 1: Check dimensions. A is 2×2, B is 2×2. Product is 2×2. [OK]

Step 2: Row 1 of A · Col 1 of B: $(1)(-1) + (2)(2) = 3$

Step 3: Row 1 of A · Col 2 of B: $(1)(0) + (2)(3) = 6$

Step 4: Row 2 of A · Col 1 of B: $(3)(-1) + (4)(2) = 5$

Step 5: Row 2 of A · Col 2 of B: $(3)(0) + (4)(3) = 12$

$$\mathbf{AB} = \begin{bmatrix} 3 & 6 \\ 5 & 12 \end{bmatrix}$$

Problem 8: Find CD.

Quiz 2: Determinants, Inverses & Systems

1. Find the determinant of [4, −3 ; 2, 1].

2. Find the inverse of [3, 1 ; 5, 2].

3. Solve using Cramer's Rule: $x - 2y = 3$, $3x + y = 2$.

4. Write the augmented matrix for: $x + z = 5$, $2x + y = 0$, $y - z = 1$.

Comprehensive Unit Test

Part A: Multiple Choice

1. If matrix A is 3×2 and matrix B is 2×4, what are the dimensions of AB?
a) 3×4 b) 2×2 c) 4×3 d) Undefined

2. If det(A) = 0, then matrix A is:
a) Invertible b) Singular c) Identity d) Symmetric

Part B: Short Answer

3. Find 3A − 2B, where A = [−1, 4 ; 0, 2] and B = [2, 0 ; −3, 1].

4. Calculate the determinant of M = [2, 1, −1 ; 0, 3, 4 ; −2, 5, 6].

5. Find the inverse of P = [7, 3 ; 2, 1]. Verify that $PP^{-1} = I$.

Part C: Extended Response

6. Solve the following system completely using Gaussian Elimination (RREF). Label all row operations.
$x + 2y - z = 3$
$2x + 5y + z = 8$
$-x + y + 3z = -1$

7. A company produces Product X and Product Y. Product X requires 2 hours of machining and 1 hour of finishing. Product Y requires 1 hour of machining and 3 hours of finishing. Machining costs $15/hr, and finishing costs $20/hr. Use matrix multiplication to find the total cost to produce one unit of Product X and one unit of Product Y.

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