

Rational & Irrational Numbers

Complete Unit — Grade 8

A Comprehensive Resource covering Definitions, Decimal Expansions, Proof by Contradiction, Rational Approximations, and Real-World Connections.

Aligned to CCSS Standards:

8.NS.A.1: Know that numbers that are not rational are called irrational.

8.NS.A.2: Use rational approximations of irrational numbers.

UM Printables

Topic 1: Rational Numbers & Decimal Expansions

CCSS 8.NS.A.1

Guided Notes

A rational number is any number that can be written as a fraction a/b , where a and b are integers and $b \neq 0$.

Rational Number: A number that can be expressed as a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Every rational number has a decimal expansion that either terminates (ends in zeros) or repeats (a pattern recurs infinitely).

Write the two types of decimal expansions for rational numbers:

Example 1: Terminating Decimal via Long Division

Convert $3/8$ to a decimal using long division.

Step 1: Set up long division: 8 outside, 3.000 inside.

Step 2: 8 goes into 30 three times ($8 \times 3 = 24$). Subtract: $30 - 24 = 6$.

Step 3: Bring down 0 to make 60. 8 goes into 60 seven times ($8 \times 7 = 56$). Subtract: $60 - 56 = 4$.

Step 4: Bring down 0 to make 40. 8 goes into 40 exactly five times ($8 \times 5 = 40$). Subtract: $40 - 40 = 0$.

Conclusion: Since we reached 0, the decimal terminates. $3/8 = 0.375$

Example 2: Repeating Decimal via Long Division

Convert $2/3$ to a decimal using long division.

Step 1: Set up long division: 3 outside, 2.000 inside.

Step 2: 3 goes into 20 six times ($3 \times 6 = 18$). Subtract: $20 - 18 = 2$.

Step 3: Bring down 0 to make 20 again. The pattern repeats indefinitely.

Conclusion: The digit 6 repeats forever: $2/3 = 0.666\ldots$

Converting Repeating Decimals to Fractions: The Algebraic Method

We can use algebra to turn a repeating decimal back into a fraction. This proves that every repeating decimal is a rational number.

Example 3: Single Repeating Digit

Convert $x = 0.333\ldots$ to a fraction.

Step 1: Let $x = 0.333\ldots$

Step 2: There is 1 repeating digit, so multiply both sides by $10^1 = 10$.

$$10x = 3.333\ldots$$

Step 3: Subtract the original equation from the new equation.

$$10x = 3.333\ldots$$

$$- x = 0.333\ldots$$

$$9x = 3$$

Step 4: Solve for x .

$$x = 3/9 = 1/3$$

Example 4: Two Repeating Digits

Convert $x = 0.454545\ldots$ to a fraction.

Step 1: Let $x = 0.454545\ldots$

Step 2: There are 2 repeating digits (45), so multiply both sides by $10^2 = 100$.

$$100x = 45.4545\ldots$$

Step 3: Subtract the original equation.

$$100x = 45.4545\ldots$$

$$- x = 0.4545\ldots$$

$$99x = 45$$

Step 4: Solve for x .

$$x = 45/99 = 5/11$$

Example 5: Non-Repeating Part Followed by Repeating Part

Convert $x = 0.1666\ldots$ to a fraction.

Step 1: Let $x = 0.1666\ldots$

Step 2: Multiply by 10 to move the decimal past the non-repeating part (1).

Proof by Contradiction: Why $\sqrt{2}$ is Irrational

How do we know for sure that $\sqrt{2}$ cannot be written as a fraction? Mathematicians use a technique called Proof by Contradiction. We assume the opposite of what we want to prove, and then show that this assumption leads to an impossible contradiction.

Full Step-by-Step Proof

Goal: Prove that $\sqrt{2}$ is irrational.

Step 1 (Assume the opposite): Assume $\sqrt{2}$ is rational. This means it can be written as a fraction a/b in simplest form, where a and b are integers with no common factors other than 1.

$$\sqrt{2} = a/b$$

Step 2 (Square both sides): Remove the square root by squaring both sides of the equation.

$$2 = a^2/b^2$$

Step 3 (Rearrange): Multiply both sides by b^2 to isolate a^2 .

$$2b^2 = a^2$$

Step 4 (Deduce properties of a): Since a^2 equals 2 times an integer (b^2), a^2 must be an even number. The only way a square is even is if the original number was even. Therefore, a must be an even number.

Step 5 (Substitute for a): Because a is even, we can write it as 2 times some integer k . Let $a = 2k$.

Substitute this back into the equation from Step 3: $2b^2 = (2k)^2$

Simplify the right side: $2b^2 = 4k^2$

Step 6 (Solve for b^2): Divide both sides by 2.

$$b^2 = 2k^2$$

Step 7 (Deduce properties of b): By the exact same logic as Step 4, since b^2 equals 2 times an integer (k^2), b^2 must be even. Therefore, b must be an even number.

Step 8 (The Contradiction!): We have proven that a is even, and b is even. This means both a and b share a common factor of 2. This directly contradicts our initial assumption in Step 1 that a/b was in simplest form!

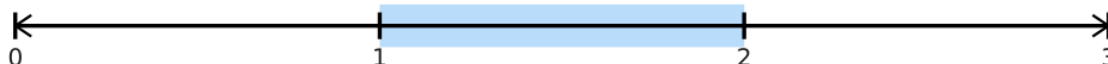
Conclusion: Since our initial assumption leads to an impossible contradiction, the assumption must be false. Therefore, $\sqrt{2}$ cannot be written as a fraction, meaning it is irrational.

Common Mistake: A common error is forgetting that a and b must be in simplest form (no common factors). This is the key assumption that makes the contradiction work. Without it, both a and b being even would not be a problem.

Visualizing $\sqrt{2}$ on Number Lines

As we narrow down the location of $\sqrt{2}$, we “zoom in” on the number line. Notice how the shaded range gets smaller and smaller.

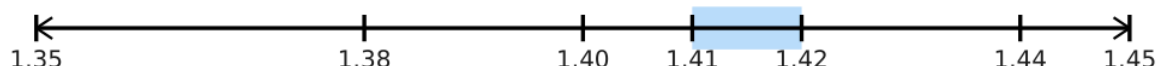
Zoom Level 1: Between 1 and 2



Zoom Level 2: Between 1.4 and 1.5



Zoom Level 3: Between 1.41 and 1.42



Comparing and Ordering Irrational Numbers

To compare irrational numbers, or to place them in order on a number line, we must first find their rational approximations.

Example 2: Ordering a Mixed Set of Numbers

Order the following numbers from least to greatest: $\sqrt{5}$, 2.3, π , $\sqrt{3}$

Step 1: Approximate all numbers to the same decimal place (hundredths).

- $\sqrt{5}$: We know $2^2 = 4$ and $3^2 = 9$. Try 2.2: $2.2^2 = 4.84$. Try 2.3: $2.3^2 = 5.29$. So $\sqrt{5} \approx 2.24$
- 2.3: This is already rational. (2.30)
- π : We know $\pi \approx 3.14$
- $\sqrt{3}$: We know $1^2 = 1$ and $2^2 = 4$. Try 1.7: $1.7^2 = 2.89$. Try 1.8: $1.8^2 = 3.24$. So $\sqrt{3} \approx 1.73$

Step 2: Write the list of approximations: 1.73, 2.24, 2.30, 3.14

Step 3: Match the approximations to the original numbers and order them.

Least to Greatest: $\sqrt{3}$, $\sqrt{5}$, 2.3, π

Estimating Expressions with Irrational Numbers

Topic 4: Real-World Connections

CCSS 8.NS.A.2

Guided Notes

Irrational numbers are not just abstract concepts; they appear naturally in the world around us, especially in geometry and science. When we calculate with irrational numbers, we must decide whether an exact answer (using symbols like π or $\sqrt{2}$) or an approximate answer (using decimals) is more appropriate.

Exact Answer: A value written in an exact form, such as 4π , $\sqrt{5}$, or $1/2$.
Approximate Answer: A rounded decimal, such as 12.57 or 2.24.

Where Irrational Numbers Appear

1. Geometry: Circles and Pi (π)

Whenever a circle is involved, π appears in standard formulas: $C = \pi d$ and $A = \pi r^2$. With a nonzero rational radius or diameter, the resulting exact circumference or area is irrational.

2. Science: Irrational Constants and the Golden Ratio

Some science formulas use irrational constants. For a simple pendulum, the idealized period is $T = 2\pi\sqrt{L/g}$, which includes π . The Golden Ratio, $(1+\sqrt{5})/2 \approx 1.618$, is used in mathematics and design and is sometimes discussed in connection with patterns in nature.

3. Diagonals of Squares

If you draw a diagonal inside a square with side length 1, the diagonal is exactly $\sqrt{2}$. Square and rectangle diagonals can be irrational. Screens are commonly specified by diagonal length, although their dimensions do not have to be based on $\sqrt{2}$.

Drill: Rational vs. Irrational Classification & Decimal Conversion

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Part A: Classify each number as Rational (R) or Irrational (I).

- | | | |
|------------------|--------------------------|--------------------|
| 1. $\sqrt{49}$ | 6. -5 | 11. 0 |
| 2. $\sqrt{50}$ | 7. 0.1010010001... | 12. e |
| 3. 0.333... | 8. 1.24444... | 13. $-\frac{2}{3}$ |
| 4. $\frac{1}{4}$ | 9. $\sqrt{\frac{25}{4}}$ | 14. 0.5 |
| 5. π | 10. $\sqrt{75}$ | 15. $\sqrt{10}$ |

Part B: Convert the following fractions to decimals.

- | | |
|-------------------|--------------------|
| 16. $\frac{3}{4}$ | 21. $\frac{1}{6}$ |
| 17. $\frac{2}{3}$ | 22. $\frac{3}{11}$ |
| 18. $\frac{5}{8}$ | 23. $\frac{9}{10}$ |
| 19. $\frac{7}{9}$ | 24. $\frac{5}{12}$ |
| 20. $\frac{4}{5}$ | 25. $\frac{7}{8}$ |

Part C: Convert the following repeating decimals to fractions in simplest form.

- | | |
|--------------|----------------|
| 26. 0.444... | 31. 0.1666... |
| 27. 0.777... | 32. 0.25555... |

Key Concepts Summary

Study Guide

1. Rational Numbers

A rational number can be expressed as a/b where a and b are integers and b does not equal 0. Rational numbers have decimal expansions that either terminate (like $3/8 = 0.375$) or repeat (like $2/3 = 0.6$ repeating). All integers, fractions, and terminating or repeating decimals are rational.

2. Irrational Numbers

An irrational number cannot be expressed as a ratio of two integers. Its decimal expansion is both non-terminating and non-repeating. Common examples include π , the square root of 2, the square root of 3, and e (Euler's number). The square root of any positive integer that is not a perfect square is irrational.

3. The Real Number System

Real numbers consist of all rational and irrational numbers. Every real number is either rational or irrational, never both. The sets nest as follows: Natural numbers are a subset of integers, which are a subset of rational numbers, which are a subset of real numbers. Rational and irrational numbers are interspersed throughout the real number line.

4. Converting Repeating Decimals to Fractions

Use the algebraic method: (1) Let x equal the repeating decimal. (2) Shift past any non-repeating digits with 10^m , then multiply by 10^n more so the repeating blocks align, where n is the number of repeating digits. (3) Subtract the equations to eliminate the repeating part. (4) Solve for x and simplify.

5. Rational Approximations

To approximate an irrational square root, find the two perfect squares it falls between, then narrow down decimal place by decimal place. For example, the square root of 2 is between 1 and 2, then between 1.4 and 1.5, then between 1.41 and 1.42. Use these approximations to compare, order, and estimate expressions involving irrational numbers.

6. Exact vs. Approximate Answers

8.NS.A.1

8.NS.A.2

Grade 8

What's Inside the Full Resource

Complete Unit — 39 Premium Pages

- 1 Rational Numbers** — Definitions, terminating & repeating decimals, long division examples
- 2 Converting Repeating Decimals** — Full algebraic method with 3 worked examples
- 3 Irrational Numbers** — Key examples table ($\sqrt{2}$, $\sqrt{3}$, π , e), decimal properties
- 4 Proof by Contradiction** — 8-step proof that $\sqrt{2}$ is irrational
- 5 Real Number System** — Venn diagram ($\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$) with examples
- 6 Successive Approximation** — Zoom-in number lines for $\sqrt{2}$
- 7 Comparing & Ordering** — Mixed sets, estimation of expressions
- 8 Real-World Problems** — 10 problems: circles, squares, ladders, pendulums
- 9 Practice Drills** — 35 classification + 25 approximation problems with answer lines
- 10 Key Concepts Summary** — Full study guide, Common Mistake tips
- 11 Assessments** — Unit Quiz + Comprehensive Test with scoring rubric

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